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# SEPARATED FLOW MODELS—I

## ANALYSIS OF THE AVERAGED AND LOCAL INSTANTANEOUS FORMULATIONS

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**Abstract**—Procedures for deriving averaged conservation equations and jump conditions for two-phase flow are discussed. Averaged working equations for stratified horizontal flow are derived and analyzed to illustrate that interfacial configuration, or flow regimes, must be considered in order to avoid inconsistencies in the model. The corresponding local instantaneous two-dimensional equations are also analyzed for propagation of disturbances in stratified flow. It is shown that the linear stability conditions for long waves are the same for both the averaged and local instantaneous cases. However, for finite amplitude waves the local instantaneous formulation leads to higher order dispersion terms that do not appear to arise in the averaged equations. In particular it is shown that finite amplitude waves are described by forms of the nonlinear Korteweg–deVries equation that can give rise to waves of permanent shape, and have fairly general classes of exact solutions.

## 1. INTRODUCTION

This is the first of a series of papers that investigates the formulation of models for transient two-phase flow. Problems of this type are of interest in the design, control and analysis of many process and power systems, such as nuclear, fossil fired and geothermal power stations. The presence of moving internal interfaces in two-phase flow makes predictions of flow behaviour much more difficult than in single phase flow. This is because the shape and movement of the interfaces are not known *a priori*, but form part of the problem being solved. Furthermore, they affect the structure of the flow field profoundly and may dominate transport processes between phases, and even at stationary boundaries.

In principle, the local instantaneous conservation equations for each phase may be written down together with appropriate molecular transport properties, initial and interfacial/wall boundary conditions. The resulting initial moving boundary value problem is intractable except in the simplest cases. Also, for some design and analysis applications, the local instantaneous behaviour of the various flow variables is not required, and predictions of averaged quantities appear to be sufficient. This is true in the many practical situations in which temporal and/or spatial fluctuations are much smaller than the average (i.e. the noise is much less than the signal and one can be clearly distinguished from the other).

Since averaged quantities are of engineering interest, one of the main approaches to two-phase flow modelling has been to average (in time, space, over an ensemble, or in some combination of these) the original local instantaneous conservation equations (e.g. Agee *et al.* 1978; Boure *et al.* 1975; Delhaye & Achard 1976; Hughes *et al.* 1976; Ishii 1975; Lyczkowski *et al.* 1978; Nigmatulin 1978; Panton 1968; Vernier & Delhaye 1968; Yadigaroglu & Lahey 1976). The resulting averaged equations and interfacial jump conditions form a mathematical model that is much simpler than the original formulation, but information is lost in the averaging process and must now be supplied in the form of auxiliary relationships. These relationships are of two types—the first for interfacial and wall transfer of mass, heat and momentum, the second for intraphase distributions of the dependent variables. To illustrate the types of auxiliary relationships consider the single phase momentum conservation equation averaged over a thin cross-sectional slice of a uniform duct.

$$\frac{\partial}{\partial t}\langle \rho u\rangle + \frac{\partial\langle \rho u^2\rangle}{\partial z} + \frac{\partial p}{\partial z} = \langle \rho F_z\rangle + \langle(\mathbf{n}\cdot\bar{\tau}_z)\rangle_w + \frac{\partial}{\partial z}\langle\tau_{zz}\rangle\,, \qquad [1]$$

where $\rho$ is density; $u$ is velocity; $p$ is pressure; $\mathbf{F}$ is the external force; $\bar{\bar{\tau}}$ is the stress tensor; $\mathbf{n}$ is the unit normal vector; and subscripts $z$ and $w$ denote components related to the $z$ direction and the wall. The term $\langle \mathbf{n} \cdot \bar{\tau}_z \rangle_w$ is the component of the averaged wall shear stress in the $z$ direction and is often approximated by the expression for steady flow

$$\langle \mathbf{n} \cdot \bar{\tau}_z \rangle_w = -2f\rho u^2/d_h , \qquad [2]$$

where $d_h$ is the hydraulic diameter, and $f$ is the friction factor. This is one type of auxiliary relationship needed, i.e. expressions for wall and interfacial transfer of mass, heat and momentum. We also need to separate the average of products into the product of averages of the dependent variables $\rho$ and $u$. For this we need relationships between $\langle \rho u \rangle$ and $\langle \rho \rangle \langle u \rangle$, and between $\langle \rho u^2 \rangle$ and $\langle \rho \rangle \langle u \rangle^2$. This is another type of auxiliary relationship, i.e. distribution relationships relating averages of products of the dependent variables to products of averages. In addition, a relationship is needed between the phase pressures in two-phase flow problems in order to close the set of averaged equations. Quite often it is implicitly assumed that the phase pressures are equal. As we will show in this and following papers, this assumption is not a good one if the behaviour of various disturbances (waves) is being investigated.

Though there is reasonable agreement between various formulations for the averaged two-phase flow models under identical physical situations and assumptions, as pointed out by Wallis (1976) and Yadigaroglu & Lahey (1976), there are still differences due to different ways of incorporating the various empiricisions required for the auxiliary relationships.

A review of the range of applicability of various models is contained in the paper by Banerjee & Hancox (1978) where comparisons with experiment are also presented. It is shown that for many physically important situations, the most powerful and physically appealing approach is to consider the interactions between the phases explicitly. In this paper we will not discuss the status of model-experiment comparisons, but assume that the "multifluid" approach discusssed later is necessary for certain problems (Banerjee & Hancox 1978).

Since this approach is still in a developmental stage we will first review the averaging process and derive a set of averaged equations. Separate sets of equations will be derived for each phase and coupled by interfacial transfers and jump conditions—an approach generally called "multifluid" modelling. Much of this is not really original except that particular attention will be paid to the treatment of the pressure terms, since these can often be the cause for differences in various formulations. Also the form of the pressure terms will be very important in the analysis presented in this and subsequent papers. For example, it will be shown in Parts II and III that higher order dispersion effects can arise even in the averaged formulation when the pressure differences between phases is properly taken into account.

We then examine propagation of finite amplitude waves in a particular flow regime, stratified two-phase flow in horizontal ducts using the local instantaneous inviscid flow formulation and compare the results with those derived from our averaged formulation. The rationale for doing this is to determine what, if anything, is lost in the averaging process. It is shown that higher order dispersive effects are obtained for the local instantaneous case even in inviscid flow, whereas these do not arise in the averaged multi-fluid models if relatively crude approximations are made to the phasic pressure difference terms.

The study on finite amplitude waves in stratified flow using the local instantaneous formulation is interesting not only because it identifies higher order dispersive effects that should arise naturally in averaged models, but also for the following reasons.

First, it represents a relatively novel approach to elucidating the form of the momentum interaction relationships in averaged models based on analysis of the local instantaneous formulation for highly idealized flow configurations. It is shown that non-linear dispersive waves arise and can be described by the Korteweg–deVries equation in certain cases. A variety of explicit exact solutions for non-linear partial differential equations of this type have been developed recently due to the remarkable work of Gardner *et al.* (1967) and Zakharov & Shabat (1971) (see also Whitham 1974, Miura 1976; Lax 1976). Because of this, a great deal of information can now be obtained on the nonlinear propagation of wave packets. We do not present solutions in this paper but will do so in later papers.

Second, as discussed by Taitel & Dukler (1976), one fruitful approach to the analysis of transitions between flow regimes starts from the condition of stratified flow in horizontal or near horizontal systems. Taitel and Dukler consider the mechanisms by which a change from stratified flow can be expected to take place, as well as the flow pattern that can be expected to result from the change. The mechanisms are related to growth and propagation of interfacial waves of various types some of which may be governed by the equations derived in this paper.

In this paper we will also introduce the perturbation method used in analysing the behaviour of finite amplitude waves. The method will also be used in subsequent papers for analysis of the averaged formulation.

We discuss averaged two-phase flow models first, and then analyse the propagation of finite amplitude waves in stratified flow.

## 2. AVERAGING PROCEDURES

Averaging operators have been discussed by Delhaye & Achard (1976), Ishii (1975) and Nigmatulin (1978). The commonly used averaging procedures are: (i) volume or area averaging, with no averaging in time; (ii) time averaging, with no averaging in space; (iii) ensemble averaging, with no averaging in space; (iv) ensemble/space averaging or time/space averaging.

The averaging procedure should lead to flow parameters that are continuous and have continuous first derivatives. The procedure should also separate "signal" from "noise", and result in averaged flow variables that can be measured with practical instrumentation.

There are some difficulties with the continuity of flow parameters and their first derivatives if they are time or cross-sectional area averaged. For example if we cross-sectional area average, then the first derivatives become discontinuous each time an interface becomes tangent to the cross-sectional plane. Similarly the time derivative of a point void fraction measurement becomes discontinuous, since at any instant the vapour phase is either present at a point or not present.

Therefore, double averages, time/space or ensemble/space, are usually used. To illustrate the procedure, consider volume averaged void fractions determined by trapping flow between two quick closing valves and measuring the proportion of each phase. The experiment can be repeated again and again by starting from the same initial conditions and by closing the valves after the same elapsed time. Results of some experiments by Banerjee *et al.* (1979) of this type are shown in figure 1. Clearly, the volume averaged void fraction fluctuates somewhat between successive experiments.

To resolve this difficulty, we may average the volume averaged void fractions over the ensemble of completed experiments and these results are also shown in figure 1. It is evident that after only a few experiments the ensemble average changes only slightly with additional experiments.

Most experimental measurements involve some degree of spatial averaging because of the instrumentation involved, but they can usually be made with good time response. While space/time averages are the simplest to obtain in experiments, difficulties may arise in distinguishing signal from noise in rapid transients. Space/ensemble averages are also straightforward to obtain from a set of repeated experiments. Indeed, a very extensive series of blow-down experiments has been performed in which this was done (see Premoli & Hancox 1976). However, for larger experiments, ensemble averages can only be obtained at prohibitive expense. Thus the subject of averaging procedures is by no means closed, especially for rapid transients.

In general, time/space or ensemble/space averaging has the necessary desirable properties

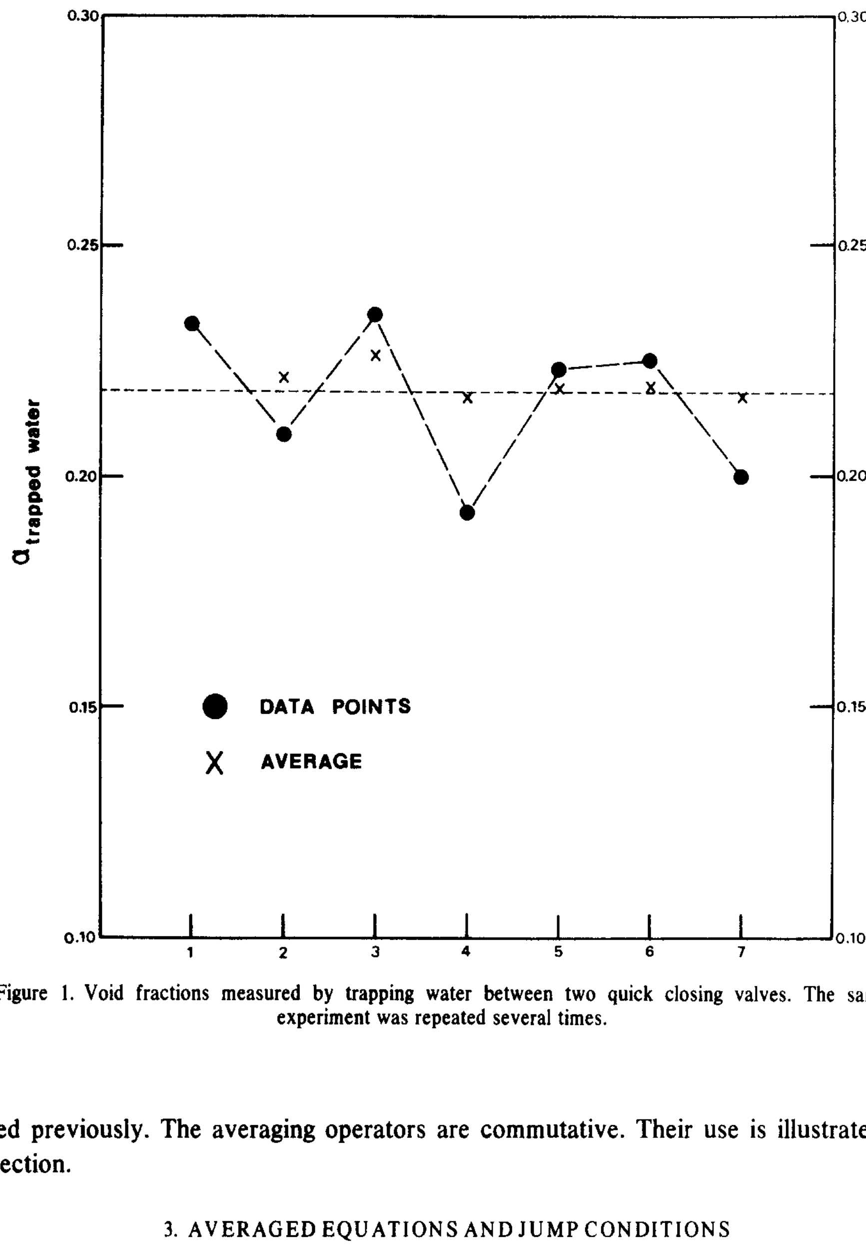


Figure 1. Void fractions measured by trapping water between two quick closing valves. The same experiment was repeated several times.

outlined previously. The averaging operators are commutative. Their use is illustrated in the next section.

## 3. AVERAGED EQUATIONS AND JUMP CONDITIONS

To illustrate the mathematical procedures used in averaging we derive the volume averaged one-dimensional conservation equations and jump conditions for transient multiphase flow in a duct of uniform cross-sectional area. Other types of averages can be derived in exactly the same way and do not change the form of the equations. In particular, time or ensemble averaging could be done first followed by volume averaging. Extensions to multidimensional flows and variable area ducts are straightforward. The resulting mathematical system derived by this approach is sometimes called a "multifluid model".

The terms that are usually most difficult to deal with appear as integrals on the r.h.s's of the equations. To derive a working set of equations these integrals are usually approximated using a set of assumptions and semi-empirical transfer relationships. It is at this point that disagreements between different investigators appear. For example, the integrals involving pressure on the r.h.s's of the momentum and energy conservation equations can be treated using a variety of different assumptions, eventually leading to differences in the derivative terms involving pressure and void fraction. This aspect will be discussed in the following development as it is a controversy that is relatively easy to resolve.

*Phase conservation equations*

Consider the flow situation in figure 2. The duct is assumed to be fixed with regard to a Galilean frame of reference. Let $V_k$ be the volume of phase $k$ enclosed between the walls and the cross-sectional planes spaced a distance $Z$ apart ($Z$ can be arbitrarily small). To derive the volume averaged form of the conservation equations, we will use Gauss' theorem and Leibnitz rule (Bird *et al.* 1960). The particular forms applying to figure 2 are given below.

As mentioned previously, the equations are derived in volume averaged form rather than cross-sectional averaged form because some difficulties arise in the latter form when the interface and the cross-section coincides as, for example, when the whole interface occupies the cross-section in a refilling problem. The distance $Z$ shown in figure 2 can be made as small as desired, so we essentially average over a slice.

The theorems we will use are given below:

*Leibnitz rule*

$$\frac{\partial}{\partial t}\int_{V_k(z,t)} f(x, y, z, t)\,\mathrm{d}V = \int_{V_k(z,t)} \frac{\partial f}{\partial t}\,\mathrm{d}V + \int_{a_i} f(\mathbf{v}_i \cdot \mathbf{n}_k)\,\mathrm{d}S. \qquad [3]$$

*Gauss' theorem*

$$\int_{V_k(z,t)} \nabla \cdot \mathbf{a}\,\mathrm{d}V = \frac{\partial}{\partial z}\int_{V_k(z,t)} \mathbf{n}_z \cdot \mathbf{a}\,\mathrm{d}V + \int_{a_i} \mathbf{n}_k \cdot \mathbf{a}\,\mathrm{d}S. \qquad [4]$$

We will define averages by the following symbols

$$\langle f_k \rangle = \frac{1}{V_k}\int_{V_k} f_k\,\mathrm{d}V, \qquad [5]$$

$$\langle f_k \rangle_i = \frac{1}{V}\int_{a_i} f_k\,\mathrm{d}S, \qquad [6]$$

where $V = \Sigma V_k$.

The local instantaneous form of the general conservation equation will not be derived as Truesdell & Toupin (1960) contains a discussion. If $\rho_k\psi_k$ is the quantity being conserved in the $k$th phase, and $\mathbf{j}_k$ and $\hat{S}_k$ are the flux and source of $\psi_k$, then

$$\frac{\partial \rho_k\psi_k}{\partial t} + \nabla \cdot \rho_k\psi_k\mathbf{v}_k + \nabla \cdot \mathbf{j}_k - \rho_k\hat{S}_k = 0.$$

The general conservation equation may be volume averaged using [3] and [4] as follows:

$$\int_{V_k} \frac{\partial \rho_k\psi_k}{\partial t}\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{V_k} \rho_k\psi_k\,\mathrm{d}V - \int_{a_i} \rho_k\psi_k(\mathbf{v}_i \cdot \mathbf{n}_k)\,\mathrm{d}S,$$

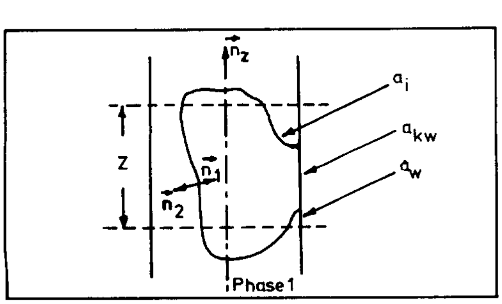


Figure 2. Definition of symbols and geometry for the averaging process.

$$\int_{V_k} \nabla \cdot (\rho_k \psi_k \mathbf{v}_k + \mathbf{j}_k)\,\mathrm{d}V = \frac{\partial}{\partial z}\int_{V_k} \mathbf{n}_z \cdot (\rho_k \psi_k \mathbf{v}_k + \mathbf{j}_k)\,\mathrm{d}V + \int_{a_i + a_{kw}} \mathbf{n}_k \cdot (\rho_k \psi_k \mathbf{v}_k + \mathbf{j}_k)\,\mathrm{d}S.$$

The usual assumption that $\mathbf{v}_k = 0$ at the wall ($a_{kw}$) has been used. The general volume averaged conservation equation is then derived as

$$\frac{\partial}{\partial t}\int_{V_k} \rho_k \psi_k\,\mathrm{d}V + \frac{\partial}{\partial z}\int_{V_k} \mathbf{n}_z \cdot (\rho_k \psi_k \mathbf{v}_k + \mathbf{j}_k)\,\mathrm{d}V - \int_{V_k} \rho_k \hat{S}_k\,\mathrm{d}V = \int_{a_i} \rho_k \psi_k \mathbf{n}_k \cdot (\mathbf{v}_i - \mathbf{v}_k)\,\mathrm{d}S - \int_{a_i + a_{kw}} \mathbf{n}_{kw} \cdot \mathbf{j}_k\,\mathrm{d}S.$$

Let $\alpha = V_k/V$ = phase volume fraction. Then using [5] (the definition of $\langle\rangle$), we have

$$\frac{\partial}{\partial t}\alpha_k\langle \rho_k \psi_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \mathbf{n}_z \cdot (\rho_k \psi_k \mathbf{v}_k + \mathbf{j}_k)\rangle + \alpha_k\langle \rho_k \hat{S}_k\rangle = -\frac{1}{V}\int_{a_i} (\dot{m}_k \psi_k + \mathbf{j}_k \cdot \mathbf{n}_k)\,\mathrm{d}S - \frac{1}{V}\int_{a_{kw}} n_{kw} \cdot \mathbf{j}_k\,\mathrm{d}S \qquad [7]$$

where we have written the interphase mass transfer rate as

$$\dot{m}_k = \rho_k \mathbf{n}_k \cdot (\mathbf{v}_k - \mathbf{v}_i). \qquad [8]$$

The forms of the conservation equations for each quantity (mass, momentum in the $z$ direction and energy) for each phase may now be derived.

*Mass.* In this case $\psi_k = 1$, $\mathbf{j}_k = 0$, $\hat{S}_k = 0$. We have

$$\frac{\partial}{\partial t}\alpha\langle \rho_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \rho_k u_k\rangle = -\langle \dot{m}_k\rangle_i = \Gamma_{mk}, \qquad [9]$$

where [6] has been used to write the r.h.s. In general, the volume averaged interfacial mass transfer rate is not known *a priori* and a correlation must be supplied.

*Linear momentum.* In this case $\psi_k = \mathbf{v}_k$, $\mathbf{j}_k = p_k\bar{\bar{I}} - \bar{\bar{\tau}}_k$, $\hat{S}_k = \mathbf{F}_K$. Taking the dot product of the conservation equation [7] with the unit $z$ direction vector $\mathbf{n}_z$, we obtain the equation for conservation of $z$ direction momentum as

$$\frac{\partial}{\partial t}\alpha_k\langle \rho_k u_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \rho_k u_k^2\rangle + \frac{\partial}{\partial z}\alpha_k\langle p_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \mathbf{n}_z \cdot (\bar{\bar{\tau}}_k \cdot \mathbf{n}_z)\rangle - \alpha_k\langle \rho_k F_{z,k}\rangle = -\frac{1}{V}\int_{a_i} [\dot{m}_k u_k + \mathbf{n}_z \cdot \mathbf{n}_k p_k - \mathbf{n}_z \cdot (\mathbf{n}_k \cdot \bar{\bar{\tau}}_k)]\,\mathrm{d}S + \frac{1}{V}\int_{a_{kw}} \mathbf{n}_z \cdot (\mathbf{n}_{kw} \cdot \bar{\bar{\tau}}_k)\,\mathrm{d}S. \qquad [10]$$

To derive [10] we must have used the relationship $\mathbf{n}_{kw} \cdot \mathbf{n}_z p_k = 0$ (since $\mathbf{n}_{kw} \cdot \mathbf{n}_z = 0$ if there is no area change). Similar equations can be derived by taking the dot product with $\mathbf{n}_x$ and $\mathbf{n}_y$. They are not shown here but can be of importance in multi-dimensional flows.

The pressure term on the r.h.s. can be further simplified. It is only singled out because some substantial simplifications are then obtained for the case of constant cross-sectional pressure. The procedure also elucidates the "apparent mass" effect. Therefore consider

$$\int_{a_i} \mathbf{n}_k \cdot (\mathbf{n}_z p_k)\,\mathrm{d}s = \int_{a_i} \mathbf{n}_k \cdot [\mathbf{n}_z(\langle p_k\rangle + \Delta p_{ki} + \Delta p'_{ki})]\,\mathrm{d}S,$$

where $\Delta p_{ki} = \langle p_{ki}\rangle - \langle p_{ki}\rangle$ is the difference between the average interfacial and average phase pressures and $\Delta p'_{ki} = p_{ki} - \langle p_{ki}\rangle$ is the difference between the local and average interfacial pressures. If $\Delta p_{ki}$ is assumed constant over $a_i$, then the fluctuating part between the bulk and interfacial pressure $\Delta p'_{ki} = 0$. However, this is only true in stratified flow with no waves. As soon as waves appear $\Delta p' \neq 0$. This is illustrated by the measurements of Miya *et al.* (1971), where the interfacial pressure is plotted as a function of position along the wave.

In addition it is possible to consider the difference between the local and average phase pressures but this does not lead anywhere because the equation is then phrased in terms of the local phase pressure and the fluctuating component. We have therefore left the momentum equation in the form shown in [12] as being the most useful.

The difference between bulk and interfacial pressures may arise for a variety of other reasons—gravitational forces in stratified flow, flow separation behind bubbles and slugs, rapid bubble growth, etc. In any case they have to be taken into account in general (as pointed out by Stuhmiller 1977). Using Gauss' theorem

$$\frac{1}{V}\int_{a_i} \mathbf{n}_k \cdot (\mathbf{n}_z p_k)\,\mathrm{d}S = -[\langle p_k\rangle + \Delta p_{ki}]\frac{\partial \alpha_k}{\partial z} + \frac{1}{V}\int_{a_i} \mathbf{n}_k \cdot (\mathbf{n}_z \Delta p'_{ki})\,\mathrm{d}S. \qquad [11]$$

Substituting this into [1], we obtain the linear momentum conservation equation as

$$\frac{\partial}{\partial t}\alpha_k\langle \rho_k u_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \rho_k u_k^2\rangle + \alpha_k\frac{\partial \langle p_k\rangle}{\partial z} - \frac{\partial}{\partial z}\alpha_k\langle \tau_{zz,k}\rangle - \Delta p_{ki}\frac{\partial \alpha_k}{\partial z} = \alpha_k\langle \rho_i F_{z,k}\rangle - \langle \dot m_k u_k\rangle_i - \langle \Delta p'_{ki}\rangle_i$$

$$+\langle\langle \mathbf{n}_k \cdot \bar\tau_z\rangle\rangle_i + \langle\langle \mathbf{n}_{kw} \cdot \bar\tau_z\rangle\rangle_w. \qquad [12]$$

Note that the equation reduces to the usual constant phase pressure form if $\Delta p_{ki} = \Delta p'_{ki} = 0$.

In the form in [12], the integral involving $\Delta p'_{ki}$ on the r.h.s. is precisely the term that leads to the "apparent mass" effect for inviscid flows. It may be calculated for inviscid flows if the shape of the interface is known *a priori*. However, this is generally not the case, and semi-empirical expressions are needed at present.

*Energy*. In this case

$$\psi_k = E_k = \left(e_k + \frac{\mathbf{v}_k \cdot \mathbf{v}_k}{2}\right); \quad \mathbf{j}_k = \mathbf{q}_k - (p_k\bar{\bar I} - \bar{\bar\tau}_k)\cdot \mathbf{v}_k; \quad \hat S_k = \mathbf{F}_k \cdot \mathbf{v}_k + Q_k.$$

Quite often $\mathbf{v}_k \cdot \mathbf{v}_k \cong u_k^2$, if the flow is dominant in the $z$ direction. We have:

$$\frac{\partial}{\partial t}\alpha_k\langle \rho_k E_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle \rho_k E_k u_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle q_{z,k}\rangle + \frac{\partial}{\partial z}\alpha_k\langle \rho_k u_k\rangle - \frac{\partial}{\partial z}\alpha_k\langle \mathbf{n}_z \cdot (\bar{\bar\tau}_k \cdot \mathbf{v}_k)\rangle = \alpha_k\langle \rho_k(\mathbf{F}_k \cdot \mathbf{v}_k + Q_k)\rangle$$

$$-\frac{1}{V}\int_{a_i} [\dot m_k E_k + \mathbf{n}_k(\mathbf{q}_k + p_k\mathbf{v}_k - \bar{\bar\tau}_k \cdot v_k)]\,\mathrm{d}S - \frac{1}{V}\int_{a_{kw}} \mathbf{n}_{kw} \cdot \mathbf{q}_k\,\mathrm{d}S. \qquad [13]$$

The equation can be written in enthalpy form as

$$E_k = h_k + \frac{v_k^2}{2} - p_k/\rho_k.$$

Now $\dot m_k = \rho_k(\mathbf{v}_k - \mathbf{v}_i)\cdot \mathbf{n}_k$ by definition

$$\therefore \quad \mathbf{n}_k \cdot p_k\mathbf{v}_k = \frac{p_k}{\rho_k}\{\mathbf{n}_k \cdot (\mathbf{v}_k - \mathbf{v}_i)\rho_k\} + p_k\mathbf{n}_k \cdot \mathbf{v}_i$$

$$= \dot m_k\frac{p_k}{\rho_k} + p_k\mathbf{n}_k \cdot \mathbf{v}_i,$$

or assuming that $\mathbf{v}_k \cdot \mathbf{v}_k = u_k^2$ we have

$$-\frac{1}{V}\int_{a_i}(\dot{m}_k E_k + \mathbf{n}\cdot p_k \mathbf{v}_k)\,\mathrm{d}S = -\frac{1}{V}\int_{a_i}\left[\dot{m}_k\left(h_k + \frac{u_k^2}{2}\right) + p_k(\mathbf{n}_k\cdot\mathbf{v}_i)\right]\mathrm{d}S. \qquad [14]$$

For consistency in treatment with regard to the momentum equation we may write

$$\frac{1}{V}\int_{a_i} p_k(\mathbf{n}_k\cdot\mathbf{v}_i)\,\mathrm{d}S = \frac{1}{V}\int_{a_i}[\langle p_k\rangle + \Delta p_{ki} + \Delta p_{ki}](\mathbf{n}_k\cdot\mathbf{v}_i)\,\mathrm{d}S,$$

as discussed previously. Using Leibnitz's rule, we find

$$\frac{1}{V}\int_{a_i} p_k(\mathbf{n}_k\cdot\mathbf{v}_i)\,\mathrm{d}s = [\langle p_k\rangle + \Delta p_{ki}]\frac{\partial\alpha_k}{\partial t} + \frac{1}{V}\int_{a_i}\Delta p'_{ki}(\mathbf{n}_k\cdot\mathbf{v}_i)\,\mathrm{d}S. \qquad [15]$$

Substituting [15] into [14] and then substituting [14] into [13], we obtain the form of the energy equation given below. Note again that a term involving $\Delta p_{ki}$ has appeared on the l.h.s. to account for the difference between the bulk and interfacial pressure. Also a term involving $\Delta p'_{ki}$ has appeared on the r.h.s.—this term is usually very small compared to those involving $\mathbf{q}_k$ and $h_k$.

$$\frac{\partial}{\partial t}\alpha_k\langle\rho_k E_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle\rho_k E_k u_k\rangle + \frac{\partial}{\partial z}\alpha_k\langle q_{z,k}\rangle + \langle p_k\rangle\frac{\partial\alpha_k}{\partial t} + \Delta p_{ki}\frac{\partial\alpha_k}{\partial t} + \frac{\partial}{\partial z}\alpha_k p_k u_k - \frac{\partial}{\partial z}\alpha_k\langle\mathbf{n}_z\cdot(\bar{\bar{\tau}}_k\cdot\mathbf{v}_k)\rangle$$

$$= -\left\langle\left[\dot{m}_k\left(h_k + \frac{u_k^2}{2}\right) + \mathbf{n}_k\cdot\mathbf{v}_i\Delta p'_{ki} + \mathbf{n}_k\cdot\mathbf{q}_k - \mathbf{n}_k\cdot\mathbf{v}_k\cdot\bar{\bar{\tau}}_k\right]\right\rangle_i - \langle(\mathbf{n}_{kw}\cdot\mathbf{q}_k)\rangle_w + \alpha_k\langle(\rho_k\mathbf{v}_k\cdot\mathbf{F}_k + Q_k)\rangle.$$

The l.h.s. may be written in enthalpy form

$$\frac{\partial}{\partial t}\alpha_k\left\langle\rho_k\left(h_k + \frac{u_k^2}{2}\right)\right\rangle + \frac{\partial}{\partial z}\alpha_k\left\langle\rho_k u_k\left(h_k + \frac{u_k^2}{2}\right)\right\rangle - \alpha_k\frac{\partial\langle p_k\rangle}{\partial t}$$

$$+ \Delta p_{ki}\frac{\partial}{\partial t}\alpha_k + \frac{\partial}{\partial z}\alpha_k\langle q_{z,k}\rangle - \frac{\partial}{\partial z}\alpha_k\langle\mathbf{n}_z\cdot(\bar{\bar{\tau}}_k\cdot\mathbf{v}_k)\rangle \qquad [16]$$

$$= -\left\langle\left[\dot{m}_k\left(h_k + \frac{u_k^2}{2}\right) + \mathbf{n}_k\cdot\mathbf{v}_i\Delta p'_{ki} + \mathbf{n}_k\cdot\mathbf{q}_k - \mathbf{n}_k\cdot\mathbf{v}_k\cdot\bar{\bar{\tau}}_k\right]\right\rangle_i - \langle(\mathbf{n}_{kw}\cdot\mathbf{q}_k)\rangle_w$$

$$+ \alpha_k\langle(\rho_k\mathbf{v}_1\cdot\mathbf{F}_k + Q_k)\rangle.$$

*Interface jump conditions*

If the interface is treated as a contact discontinuity the local instantaneous form of the generalized conservation equation across the interface (i.e. jump condition) is

$$\sum_{k=1}^{2}[\rho_k\psi_k(\mathbf{v}_k - \mathbf{v}_i) + \mathbf{j}_k]\cdot\mathbf{n}_k = 0,$$

or

$$\sum_{k=1}^{2}(\dot{m}_k\psi_k + \mathbf{j}_k\cdot\mathbf{n}_k) = 0. \qquad [17]$$

These local instantaneous equations may be volume averaged in the same way as the phase conservation equations.

*Interface mass*

$$\sum_{k=1}^{2}\frac{1}{V}\int_{a_i}\dot{m}_k\,\mathrm{d}V=0,$$

or

$$\langle\dot{m}_1\rangle_i=-\langle\dot{m}_2\rangle_i. \qquad [18]$$

*Interface linear momentum*

$$\sum_{k=1}^{2}\frac{1}{V}\int_{a_i}[\dot{m}_k u_k+\mathbf{n}_z\cdot(\mathbf{n}_k p_k)-\mathbf{n}_z\cdot(\mathbf{n}_k\cdot\bar{\bar{\tau}}_k)]\,\mathrm{d}S=0.$$

Using [11] and [18], and the identity $\mathbf{n}_1=-\mathbf{n}_2$

$$[\langle p_{1i}\rangle-\langle p_{2i}\rangle]\frac{\partial\alpha}{\partial z}=\langle\dot{m}_1(u_1-u_2)+(\Delta p'_{1i}-\Delta p'_{2i})-\mathbf{n}_1\cdot(\bar{\tau}_{1,z}-\bar{\tau}_{2,z})\rangle_i,$$

where $\alpha=\alpha_1$ and $\alpha_2=1-\alpha$.

*Interface energy*

$$\sum_{k=1}^{2}\int_{a_i}\left[\dot{m}_k\left(h_k+\frac{u_k^2}{2}\right)+p_k(\mathbf{n}_k\cdot\mathbf{v}_i)+n_k\cdot(\mathbf{q}_k-\bar{\bar{\tau}}_k\cdot\mathbf{v}_k)\right]\mathrm{d}S=0.$$

Using [15] and [18], we have

$$[\langle p_{1i}\rangle-\langle p_{2i}\rangle]\frac{\partial\alpha}{\partial t}=\Big\langle\mathbf{n}_1\cdot[(\mathbf{v}_1\cdot\bar{\bar{\tau}}_1)-(\mathbf{v}_2\cdot\bar{\bar{\tau}}_2)]-\mathbf{n}_1\cdot(\mathbf{q}_1-\mathbf{q}_2)-\mathbf{n}_1\cdot\mathbf{v}_i(\Delta p'_{1i}-\Delta p'_{2i})$$
$$-\dot{m}_1\left[\left(h_1+\frac{u_1^2}{2}\right)-\left(h_2+\frac{u_2^2}{2}\right)\right]\Big\rangle_i.$$

Note again that the $\Delta p'_{ki}$ terms could be very small compared to $\mathbf{q}_k$ and $h_k$.

All the equations in this section have been derived in instantaneous volume averaged form. They can now be time or ensemble averaged and the form remains exactly the same. Terms like $(\partial\alpha_k\langle\rho_k u_k\rangle/\partial t)$ become $\overline{(\partial\alpha_k\langle\rho_k u_k\rangle/\partial t)}$ where the overbar denotes time or ensemble averaging. The averaging operators are commutative.

*Comments regarding phasic conservation equations and jump conditions*

*Treatment of terms involving pressure.* The one dimensional form of the volume averaged phasic conservation equations are given by [9], [10] or [12], [13] or [16]. The jump conditions are given by [18], [19] and [20]. If $\Delta p_{ki}$ is not assumed constant, integrals involving the fluctuating part, $\Delta p'_{ki}$, appear on the r.h.s's of [12], [16], [19] and [20].

Many investigators assume that $p=p_1=p_2=p_{1i}=p_{2i}$. In that case the equations presented would simplify such that terms involving $\Delta p_{ki}$, $\Delta p'_{ki}$ and $(\langle p_{1i}\rangle-\langle p_{2i}\rangle)$ would vanish. Furthermore, $p_k$ would be replaced by $p$ in [12] and [16].

If pressure is assumed to be constant in the slice, then the momentum equation is given by [12] with $\Delta p_{ki}=\Delta p'_{ki}=0$, and

$$\alpha_k\frac{\partial\langle p_k\rangle}{\partial z}\to\alpha_k\frac{\partial p}{\partial z}.$$

As mentioned earlier the fluctuating part leads to derivative terms in the momentum interaction relationships due, for example, to the apparent mass effect. In general, when the interface is not aligned in the direction of mean motion, there will be an apparent mass force even in accelerating inviscid flows—therefore, this must be explicitly added to the momentum equation for all flow regimes except smooth stratified flow. The term arises naturally as part of the integrals involving the pressure interactions.

If the pressure is not assumed constant but is left as an integral on the r.h.s. as in [10], then the l.h.s. will contain

$$\frac{\partial \alpha_k \langle p_k \rangle}{\partial z}.$$

There is nothing incompatible between the forms of [10] and [12]. In [12] the pressure interaction has been made as explicit as possible and therefore it is to be preferred, provided an integral involving $\Delta p'_{ki}$ is added to the r.h.s. This integral must be evaluated for various flow regimes to at least obtain the apparent mass term. Smooth stratified flow is the only flow regime in which this term vanishes.

Turning now to the jump conditions, we see that terms containing the derivatives of $\alpha$ and $(p_{1i} - p_{2i})$ occur in the momentum and energy conditions [19] and [20]. In flows with no (or low) mass transfer (and neglecting surface tension) $\langle p_{1i} \rangle - \langle p_{2i} \rangle$ is usually assumed to vanish and these jump conditions simplify to their usual form. However in cases where there is a high rate of mass transfer between phases (due to rapid vaporisation or condensation) and large gradients in $\alpha$, the l.h.s.'s of [19] and [20] may be significant. In this case $\Delta p_{ki}$ may be small, but $\langle p_{1i} \rangle - \langle p_{2i} \rangle$ may be large. This is particularly true for flows where the mass transfer induced interfacial forces are much larger than the other interfacial forces (e.g. rapid condensation in stratified flow).

*Distribution effects in the derivative terms.* To obtain a working set of equations, the averages of products of the dependent variables must be related to the product of averages. The assumption is generally made that phase density variations within the averaging volume are small so that distribution effects for quantities like $\overline{\rho_k}\ \overline{\alpha_k \langle u_k \rangle}$, $\overline{\rho_k}\ \overline{\alpha_k \langle u_k^2 \rangle}$ have to be considered.

It is usual to define average velocities and enthalpies as

$$\overline{\langle u_k \rangle_k} \triangleq \frac{\overline{\alpha_k \langle u_k \rangle}}{\bar{\alpha}_k} \quad [22]$$

$$\overline{\langle h_k \rangle_k} \triangleq \frac{\overline{\alpha_k \langle h_k \rangle}}{\bar{\alpha}_k}. \quad [23]$$

See also Yadigaroglu & Lahey (1976) for a discussion. Definition [22] allows the mass conservation equation [9] to be written in terms of $\overline{\alpha_k}$, $\overline{\rho_k}$ and $\overline{\langle u_k \rangle_k}$. However for the momentum and energy equations, we require to define various distribution coefficients. Then the momentum equation [12] requires distribution coefficients of the form

$$C_k \triangleq \frac{\overline{\alpha_k \langle u_k^2 \rangle}}{\overline{\alpha_k}\,\overline{\langle u_k \rangle_k}^2}, \quad [24]$$

and the energy equation [16] requires

$$C'_k \triangleq \frac{\overline{\alpha_k \langle h_k u_k \rangle}}{\overline{\alpha_k}\,\overline{\langle h_k \rangle_k}\,\overline{\langle u_k \rangle_k}}, \quad [25]$$

$$C''_k \triangleq \frac{\overline{\alpha_k \langle u_k^3 \rangle}}{\overline{\alpha_k}\,\overline{\langle u_k \rangle_k}^3}. \quad [26]$$

The time or ensemble and volume averaging symbols in definitions [22]–[26] may be interchanged. Note that the distribution coefficients are not Galilean invariant.

If the forms of these distribution coefficients are known then the l.h.s's of the conservation equations can be written in terms of $\overline{\rho_k}$, $\overline{\alpha_k}$, $\langle\overline{u_k}\rangle_k$ and $\langle\overline{h_k}\rangle_k$. At the present stage of knowledge, very little data on distribution coefficients are available, so they are generally set equal to unity. It is possible to evaluate these coefficients taking power law profiles like those suggested by Bankoff (1960).

To illustrate the magnitude of distribution effects we have calculated $C_k$ (using [24]) for the power law velocity and phase volume fraction profiles shown in figure 3. It is evident from figure 3 that $C_k \cong 1.0$ for flat velocity profiles, and only deviates significantly from unity for very peaked velocity profiles. Therefore, it may be reasonable to put $C_k = 1.0$ if the portion of the phase being considered is relatively homogeneous. One approach in multifluid modelling has been to split a phase into two components if they are known to have very different velocities and enthalpies. Thus in annular flow the liquid may be split into a droplet "phase" and a "liquid film" phase (see Saito 1977). Instead of writing a set of liquid phase conservation equations with distribution coefficients, one may write two sets of conservation equations for the liquid phase—one set for the droplets and one set for the liquid film—with distribution coefficients set equal to unity. Similarly, for subcooled boiling, where temperature distributions in the liquid phase are important, the approach has been to account for this by splitting the vapour generation (mass transfer) source term into two components—one for the bulk liquid and one for the wall liquid (see Hughes *et al.* 1976).

This approach, of course, merely shifts the problem from one regarding distribution coefficients to one regarding transfer relationships for the new "phase(s)". However, in cases like annular flow and subcooled boiling it appears easier to determine the transfer relationships for a new "phase" on the basis of the data available. The approach taken in future will undoubtedly depend on the flow regime being modelled, and the type of data that can be taken.

*Distribution effects in algebraic terms.* We have considered distribution effects in the derivative terms (the l.h.s's of the conservation equations). To derive a working set of equations, interphase transfer relationships phrased in terms of the averaged dependent variables $\overline{\alpha_k}$, $\overline{\rho_k}$, $\langle\overline{u_k}\rangle_k$, $\langle\overline{h_k}\rangle_k$ are also necessary. As mentioned previously, these relationships have to be supplied

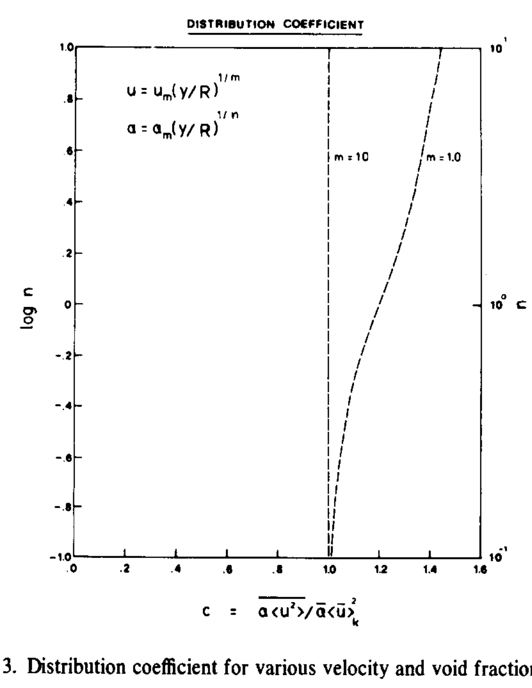


Figure 3. Distribution coefficient for various velocity and void fraction profiles.

largely on the basis of empirical information or by analogy to steady single phase flow (see Hughes *et al.* 1976 for a compilation of expressions for interphase transfers in the various flow regimes). Leaving aside effects due to large temporal and spatial gradients, the expressions are often based on relationships requiring concepts like friction factors, drag coefficients and heat transfer coefficients. Consider for example the interfacial frictional term $\langle\overline{\mathbf{n}_k \cdot \bar{\tau}_z}\rangle_i$ on the r.h.s. of [12]. This term is sometimes modelled by a frictional term of the form $\langle\overline{a_1 B_k (u_1 - u_2)}\rangle$, where $a_i$ is the local interfacial area, $B_k$ is a coefficient involving densities, perhaps local velocity differences, and a characteristic local length scale. $(\overline{u_1} - \overline{u_2})$ is the ensemble or time averaged local velocity difference. We wish, in general, to write this as $\langle\overline{a_1}\rangle\langle\overline{B_k}\rangle\langle\overline{u_1}\rangle_1 - \langle\overline{u_2}\rangle_2)$. If distribution effects in $a_i$ and $B_k$ are ignored for the moment we still have to deal with the distribution of velocities. The resulting distribution coefficient, can be calculated for various power law profiles and is not insignificant, as pointed out by Zolotar & Lellouche (1979). For constant $a_i$ and $B_k$ across the cross-section, the distribution coefficient may be directly obtained for power law profiles from the results of Bankoff (1960). The same effect obtains also for interfacial heat transfer $q_{ik}$ if it is written in terms of a local temperature difference and heat transfer coefficients.

Because of these rather complicated distribution effects that may occur in the interphase transfer relationships, it again appears physically appealing, where possible, to split phases with large cross-stream variations in velocity or temperature into two or more relatively homogeneous components.

## 4. AVERAGED WORKING EQUATIONS AND CHARACTERISTICS FOR STRATIFIED FLOW

The conservation equations [9], [12] and [16] may be written in the primitive form (see Richtmeyer & Morton 1967).

$$\bar{A}\frac{\partial \bar{U}}{\partial t} + \bar{B}\frac{\partial \bar{U}}{\partial z} = \bar{D}, \quad [27]$$

$\bar{A}$ and $\bar{B}$ are coefficient matrices, $\bar{U}$ and $\bar{D}$ are vectors containing the dependent variables and algebraic terms in the interphase transfer relationships. Much of the difficulty in multifluid modelling lies in specifying the relationships for the interphase transfer relationships in $\bar{D}$.

To close the set of equations, we require equations of state for each phase, i.e. $h_k = f(p_k, \rho_k)$, distribution and interphase transfer relationships, initial and boundary conditions. The distribution and interphase transfer relationships have to be obtained using empirical information.

To simplify the situation, consider first the simple case of incompressible inviscid flow with no derivative terms in the interphase transfer relationships and constant cross-sectional pressure. We also assume the distribution coefficients $C_k = C'_k = C''_k = 1.0$. This leads to the r.h.s. of [27] being zero. (The nearest physical situation corresponding to these assumptions is smooth stratified flow; however to model this properly, variation in cross-sectional pressure due to gravitational forces have to be taken into account, as shown later.) For the assumptions stated

$$\bar{U}^T = [u_1, u_2, h_1, h_2, p, \alpha] \quad \text{and} \quad \bar{D} = 0,$$

where the averaging signs have been dropped, i.e. $\langle\overline{u_k}\rangle_k = u_k$, $\langle\overline{h_k}\rangle_k = h_k$, etc. Also

$$\bar{A} = \begin{bmatrix} \alpha\rho_1 & & & & & \\ & (1-\alpha)\rho_2 & & & & \\ & & \alpha\rho_1 & & -\alpha & \\ & & & (1-\alpha)\rho_2 & -(1-\alpha) & \\ & & & & & \rho_1 \\ & & & & & -\rho_2 \end{bmatrix} \quad [28]$$

$$\hat{B}=\begin{bmatrix} \alpha\rho_1 u_1 & & & & \alpha & \\ & (1-\alpha)\rho_2 u_2 & & & 1-\alpha & \\ & & \alpha\rho_1 u_1 & & -\alpha u_1 & \\ & & & (1-\alpha)\rho_2 u_2 & -(1-\alpha)u_2 & \\ \alpha\rho_1 & & & & & \rho_1 u_1 \\ & (1-\alpha)\rho_2 & & & & -\rho_2 u_2 \end{bmatrix} \qquad [29]$$

The characteristic determinant is

$$\det|\tilde{B}-\lambda\tilde{A}|=0\,.$$

The characteristics are $\lambda=(\mathrm{d}z/\mathrm{d}t)=u_1,\ u_2$, two infinities corresponding to the incompressibility condition and the two roots of

$$\alpha\rho_2(u_2-\lambda)^2+(1-\alpha)\rho_1(u_1-\lambda)^2=0\,. \qquad [30]$$

The roots are real only if

$$-\alpha(1-\alpha)\rho_1\rho_2(u_1-u_2)^2\geq 0\,. \qquad [31]$$

For all two-phase flows with $u_1\neq u_2$, [31] is never satisfied. Therefore $\lambda$ has an imaginary part and instabilities can be expected.

Before proceeding further it should be recognized that the formulation in [27] is only valid for smooth stratified flow, otherwise derivative terms for at least the apparent mass effect must be included. It does not appear physically realistic that a smooth stratified flow can be maintained without a stabilizing force (say due to gravity) because of the possibility of Kelvin–Helmholtz instability. It is therefore of interest to determine the effect of a force of this type. Consider stratified flow in a rectangular channel as shown in figure 4. Appropriate expressions for $\Delta p_{ki}$ in [9], [12] and [16] are now necessary. They can be derived from the transverse momentum conservation equation but this is rather difficult if the transverse velocity field is not being determined. The simplest approach is to assume a static force balance to obtain $\Delta p_{ki}$. If we also assume no mass transfer and surface tension then $p_{1i}=p_{2i}$, and $\Delta p'_{ki}=0$ (smooth interface aligned in direction of flow), and

$$p_i-p_1=\frac{\rho_1 g\alpha H}{2}=\Delta p_{1i}\,, \qquad [32]$$

$$p_i-p_2=\frac{-\rho_2 g(1-\alpha)H}{2}=\Delta p_{2i}\,. \qquad [33]$$

These expressions can be refined further, as stated before, by considering the transverse

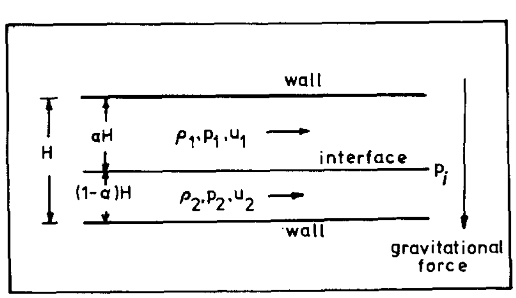


Figure 4. Definition of symbols and geometry for analysis of stratified flow using the averaged formulation.

momentum balance. Incorporating these terms into $\tilde{A}$ and $\tilde{B}$ leads to the following changed expression in place of the two complex characteristics given by [31]

$$\lambda = V_0 \pm \left[\frac{-(u_1-u_2)^2}{\alpha/\rho_1+(1-\alpha)/\rho_2}+(\rho_2-\rho_1)gH\right]^{1/2}\left[\frac{\rho_1}{\alpha}+\frac{\rho_2}{1-\alpha}\right]^{-1/2}, \qquad [34a]$$

where

$$V_0 = \frac{(1-\alpha)u_1\rho_1+\alpha\rho_2 u_2}{\alpha\rho_2+\rho_1(1-\alpha)}. \qquad [34b]$$

The characteristics are wholly real when

$$(\rho_2-\rho_1)gH\left[\frac{\alpha}{\rho_1}+\frac{(1-\alpha)}{\rho_2}\right] \geq (u_1-u_2)^2. \qquad [35]$$

It should be noted that in this case the characteristics are identical to the phase velocities ($\omega/k$) that would be obtained from a linear dispersion analysis, in which the dependent variables $\bar{U}$ are perturbed by a wave $\hat{U}$ $\exp[i(kz-\omega t)]$, and the relationship between $\omega$ and $k$ is obtained. These conditions are exactly the same as derived by Wallis (1969) and Milne-Thomson (1960) for one-dimensional gravity waves in ducts using a quite different method. Delhaye (1978) and Rousseau & Ferch (1979) appear to have first derived this result using a somewhat similar analysis. It is clear that gravity stabilizes the flow over a significant range of parameters. However, when the inequality in [35] is violated, interfacial waves would presumably appear and additional interfacial momentum interactions such as due to apparent mass effects would have to be incorporated. We shall show in the next section that [34] and [35] hold for the linearized phase velocity for long waves in stratified flow using an instantaneous two-dimensional formulation.

The effect of compressibility can also be incorporated quite simply, in which case if $(gH/c_k^2) \ll 1.0$, the characteristics become $\lambda = u_1, u_2$ the two roots given in [34], and $U_0 \pm C_0$ where

$$U_0 = \frac{(1-\alpha)\rho_1 u_2+\alpha\rho_2 u_1}{(1-\alpha)\rho_1+\alpha\rho_2}, \qquad [36a]$$

$$C_0 = \left[\frac{\dfrac{\alpha}{\rho_1}+\dfrac{1-\alpha}{\rho_2}}{\dfrac{\alpha}{\rho_1 C_1^2}+\dfrac{1-\alpha}{\rho_2 C_2^2}}\right]^{1/2}. \qquad [36b]$$

The expression for $C_0$ was also derived by Wallis (1969) using a different method and called the "stratified" sound speed. The condition for all the characteristics to be wholly real remains the same as in [35]. It should be noted at this stage that the characteristics are identical to the phase velocities arising out of a linear dispersion analysis for the case of inviscid flow with no mass transfer. A linear dispersion analysis would therefore yield the same results as the characteristics analysis. This is because $\bar{D} = 0$ for the cases considered.

We will also show in the next section that analysis of the instantaneous two-dimensional equations for stratified compressible flow yields the same linear phase velocities for long waves as given by the expressions in [34] and [36].

Thus the condition for the characteristics to be wholly real in this case appears to correspond to the linear interfacial stability condition for long waves, which is quite reasonable. When wave amplitudes become significant, momentum coupling between the phases become more complex and may contain derivative terms (due to apparent mass effects) even if we assume inviscid flow with no mass transfer. Cheng (1977) considered the general form of the apparent mass force and obtained

$$\mathbf{F}_{VM} = C_{VM}\rho_c\bar{a}_{VM}, \quad [37]$$

where

$$\mathbf{a}_{VM} = \frac{\mathrm{d}_l\mathbf{u}_g}{\mathrm{d}t} - \frac{\mathrm{d}_g\mathbf{u}_l}{\mathrm{d}t} + (1-\lambda)(\mathbf{u}_g - \mathbf{u}_l)\cdot\nabla(\mathbf{u}_g - \mathbf{u}_l), \quad [38]$$

and $\lambda$ and $C_{VM}$ may be adjusted for various flow regimes, void fractions, etc. Though Cheng did not suggest this, $C_{VM}$ is, in general, a tensor. Nigmatulin (1978) solved the problem for a particular flow configuration and obtained a somewhat different relationship for $\mathbf{a}_{VM}$, but it also only contained first derivatives of the dependent variables.

These apparent mass effects may be incorporated in the averaged working equations, and they will modify the characteristics somewhat. However, they are only first order terms and as long as we assume inviscid flow with no interfacial mass transfer, i.e. the r.h.s. of [27] is zero, the waves are purely hyperbolic and not dispersive. Certainly, no higher order dispersive effects enter [27] even when the suggested derivative terms for apparent mass effects are incorporated. (By dispersive is meant a system in which the phase velocity $\omega/k$, of a disturbance of frequency $\omega$ and wave number $k$, depends on $k$, i.e. $\mathrm{d}^2\omega/\mathrm{d}k^2 \neq 0$.)

## 5. ANALYSIS OF THE LOCAL INSTANTANEOUS FORMULATION

The local instantaneous form of the non-linear equation for bubble compression and expansion has been considered by van Wijngaarden (1968) in analysing the propagation of pressure waves through bubbly mixtures. For pressure waves moving in one direction, he showed that they could be described by the Korteweg–deVries equation. Finite amplitude waves have also been considered by Nayfeh (1976) for two, incompressible, inviscid, semi-infinite fluids. Using the method of multiple scales, he was able to describe the evolution of interfacial wave packets by two nonlinear Schrodinger equations. Finite amplitude pressure waves propagating in a gas–liquid layer at rest has been considered by Moriaka & Matsui (1975). A growing literature on wave propagation in two-phase flow has also recently appeared in the U.S.S.R. (see Kuznetsov *et al.* 1978). Much of it is concerned with bubbly flow and essentially starts with the homogeneous two-phase flow equations. The relationship of this work to multifluid models is discussed in Part III. Thus precedence exists in somewhat different situations for the type of investigation that we will now describe.

### *Incompressible stratified flow*

We will start with the analysis for stratified flow of two incompressible fluids in a horizontal duct of height $H$ as shown in figure 5. (Note that the coordinate in the flow direction is now designated by $x$.) We will later extend the discussion to compressible flow. It is easier to handle

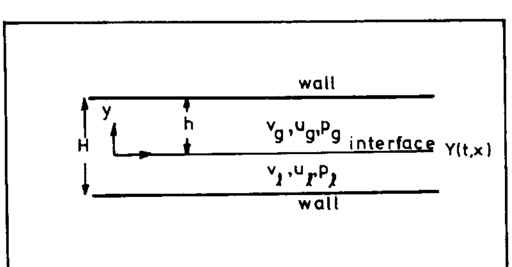


Figure 5. Definition of symbols and geometry for analysis of stratified flow using local instantaneous two-dimensional formulation.

the problem in the primitive form of the equation for the compressible case. Therefore we deal with the primitive form of the equations for the incompressible case as well. The conservation equations are:

$$\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y} = 0, \qquad [39]$$

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + v_i\frac{\partial u_i}{\partial y} = -\frac{1}{\rho_i}\frac{\partial P_i}{\partial x}, \qquad [40]$$

$$\frac{\partial v_i}{\partial t} + u_i\frac{\partial v_i}{\partial x} + v_i\frac{\partial v_i}{\partial y} = -\frac{1}{\rho_i}\frac{\partial P_i}{\partial y} - g. \qquad [41]$$

With the boundary conditions:

$$v_g = 0 \quad \text{at} \quad y = h, \qquad v_l = 0 \quad \text{at} \quad y = -(H-h),$$

$$P_g = P_l + T\frac{\partial^2 Y}{\partial x^2}, \qquad v_i = \frac{\partial Y}{\partial t} + u_i\frac{\partial Y}{\partial x} \quad \text{at} \quad y = Y(t,x), \qquad [42]$$

where $Y(t,x)$ describes the interface. Note that the boundary conditions are only correct for small displacements; otherwise the expression for the radius of curvature would be more complex. Unperturbed flow velocities are $u_i(0)$ in the $x$ direction (see figure 5).

The unperturbed interface is at $y = 0$, and the subscript $i$ denotes the gas or liquid phase. A linear dispersion analysis indicates that the system is dispersive and suggests that the weak non-linearity may be considered for long waves by applying the reductive perturbation method (Jeffrey & Kakutani 1972). Because of the complexity of the derivation we will sketch the method here.

We take the stretched coordinates

$$\tau = \epsilon^{3/2}t, \quad \xi = \epsilon^{1/2}(x/v - t), \quad y = y,$$

where $v$ is the phase velocity to be determined from the following analysis, and $\epsilon$ is a small parameter of the order of the surface displacements, which are assumed to be small but finite. The dependent variables may be expanded as

$$\begin{aligned} u_i &= u_i^{(0)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots \\ v_i &= \epsilon^{1/2}(\epsilon v_i^{(1)} + \epsilon^2 u_i^{(2)} + \cdots \\ P_i &= P_i^{(0)} + \epsilon P_i^{(1)} + \epsilon^2 P_i^{(2)} + \cdots \\ Y &= \epsilon Y^{(1)} + \epsilon^2 Y^{(2)} + \cdots . \end{aligned} \qquad [43]$$

The heuristic basis for expansions of this type and their range of application is discussed fully in Jeffrey & Kakutani (1972) and will not be repeated here. For terms not involving $\epsilon$ we obtain the solution

$$P_i^{(0)}(y) - P_i^{(0)}(0) = \rho_i g(Y - y), \qquad [44]$$

where $P_i^{(0)}(0)$ is the unperturbed pressure at the interface and $P_i^{(0)}(y)$ is the unperturbed pressure at any position in the fluid. To the next order of $\epsilon$ we obtain

$$\frac{1}{V}\frac{\partial}{\partial \xi}u_i^{(1)} + \frac{\partial v_i^{(1)}}{\partial y} = 0, \qquad [45]$$

$$-\frac{\partial u_i^{(1)}}{\partial \xi}+\frac{u_i^{(0)}}{v}\frac{\partial u_i^{(1)}}{\partial \xi}+\frac{g}{v}\frac{\partial Y^{(1)}}{\partial \xi}+\frac{1}{\rho_i v}\frac{\partial P_i^{(1)}}{\partial \xi}=0\,, \qquad [46]$$

$$\frac{\partial P_i^{(1)}}{\partial y}=0\,, \qquad [47]$$

with the boundary conditions:

$$\begin{aligned} &P_l^{(1)}=P_g^{(1)} \quad \text{at} \quad y=0\,; \\ &v_g^{(1)}=0, \quad \text{at} \quad y=h; \quad v_l^{(1)}=0 \quad \text{at} \quad y=-(H-h)\,; \\ &v_g^{(1)}/[1-u_g^{(0)}/v]=v_l^{(1)}/[1-u_l^{(0)}/v]=-\partial Y^{(1)}/\partial\xi \quad \text{at} \quad y=0\,. \end{aligned} \qquad [48]$$

The equations may be integrated and the boundary conditions used to yield

$$P_g^{(1)}(\xi,\tau)=P_l^{(1)}(\xi,\tau)=P^1(\xi,\tau) \qquad [49]$$

at $y=0$

$$v_g^{(1)}=\{[(1-u_g^{(0)}/v)h]/[\rho_g(hg+v^2(1-u_g^{(0)}/v)]\}\partial P^{(1)}/\partial\xi\,, \qquad [50]$$

and

$$v_l^{(1)}=\{-(H-h)(1-u_l^{(0)}/v)/[\rho_l(v^2(1-u_l^{(0)}/v)^2-(H-h)g)]\}\partial P^{(1)}/\partial\xi\,. \qquad [51]$$

Using the boundary condition at $y=0$, we obtain

$$\frac{h}{\rho_g[h_g+(v-u_g^{(0)})^2]}+\frac{(H-h)}{\rho_l[(v-u_l^{(0)})^2-(H-h)g]}=0\,.$$

Writing $h/H=\alpha$, we can solve for $v$ to get

$$\mathrm{v}=\frac{\alpha\rho_l u_l^{(0)}+(1-\alpha)\rho_g u_g^{(0)}}{\alpha\rho_l+(1-\alpha)\rho_g}\pm\left[\frac{-(u_g^{(0)}-u_l^{(0)})^2}{\alpha/\rho_g+(1-\alpha)/\rho_l}+(\rho_l-\rho_g)gH\right]^{1/2}\left[\frac{\rho_g}{\alpha}+\frac{\rho_l}{1-\alpha}\right]^{-1/2}. \qquad [52]$$

This is the result obtained from the characteristics analysis of the averaged stratified flow equations given in [34]. The same stability condition as presented in [35] also holds here. The method used here is, of course, supposed to explain the evolution of an instability, though the growth rate has to be relatively slow. Note that 0(1) change in $\tau$ implies as $0(\epsilon^{-3/2})$ change in $t$.

In order to proceed to the next order in $\epsilon$ we need to obtain expressions for $Y^{(1)}$ and $u_i^{(1)}$. These are

$$P^{(1)}(\xi,\tau)=A\,Y^{(1)}(\xi,\tau)\,, \qquad [53]$$

where

$$A=\rho_l[v^2(1-u_l^{(0)}/v)^2-g(H-h)]/(H-h)-\rho_g[v^2(1-u_g^{(0)}/v)^2+gh]/h\,, \qquad [54]$$

and

$$u_i^{(1)}=[gY^{(1)}/v+P_i^{(1)}/(\rho_i v)]/(1-u_i^{(0)}/v)\,. \qquad [55]$$

Consideration of the next order terms gives

$$\frac{1}{v}\frac{\partial u_i^{(2)}}{\partial \xi}+\frac{\partial v_i^{(2)}}{\partial y}=0\,, \qquad [56]$$

$$-\frac{\partial u_i^{(2)}}{\partial \xi}+\frac{u_i^{(0)}}{v}\frac{\partial u_i^{(2)}}{\partial \xi}+\frac{1}{\rho_i v}\frac{\partial P_i^{(2)}}{\partial \xi}+\frac{g}{v}\frac{\partial Y^{(2)}}{\partial \xi}=-\frac{\partial u_i^{(1)}}{\partial \tau}-v_i^{(1)}\frac{\partial u_i^{(1)}}{\partial y}-\frac{u_i^{(1)}}{v}\frac{\partial u_i^{(1)}}{\partial \xi}\,, \qquad [57]$$

$$\frac{1}{\rho_i}\frac{\partial P_i^{(2)}}{\partial y}=\frac{\partial v_i^{(1)}}{\partial \xi}-\frac{u_1^{(0)}}{v}\frac{\partial v_i^{(1)}}{\partial \xi}\,, \qquad [58]$$

with the boundary conditions

$$v_g^{(2)}=0 \quad \text{at} \quad y=h; \qquad v_l^{(2)}=0 \quad \text{at} \quad y=-(H-h)\,,$$

$$P_g^{(2)}=P_l^{(2)}+\frac{T}{v^2}\frac{\partial^2 Y^{(1)}}{\partial \xi^2} \quad \text{at} \quad y=0\,;$$

$$\left[\frac{\partial Y^{(1)}}{\partial \tau}-v_g^{(2)}+\frac{u_1^{(0)}}{v}\frac{\partial Y^{(1)}}{\partial \xi}\right]\Big/\left[1-\frac{u_g^{(0)}}{v}\right]=\left[\frac{\partial Y^{(1)}}{\partial \tau}-V_l^{(2)}+\frac{u_l^{(1)}}{v}\frac{\partial Y^{(1)}}{\partial \xi}\right]\Big/\left[1-\frac{u_l^{(0)}}{v}\right] \qquad [59]$$

$$=\frac{\partial Y^{(2)}}{\partial \xi} \quad \text{at} \quad y=0\,.$$

The expressions on the r.h.s's of [56]–[58] are known from the previous perturbation. These expressions can be substituted and the equations integrated for $P_i^{(2)}$, $v_i^{(2)}$ and $u_i^{(2)}$. The expressions for these quantities must satisfy the boundary conditions given in [59]. On integrating [58] we get

$$P_g^{(2)}=B_g\frac{\partial^2 P^{(1)}}{\partial \xi^2}(hy-y^2/2)+F_g(\tau,\xi)\,, \qquad [60]$$

where

$$B_g=\frac{(1-u_g^{(0)}/v)^2}{[hg+v^2(1-u_g^{(0)}/v)^2}\,,$$

and

$$P_l^{(2)}=B_l[(H-h)y+y^2/2]\frac{\partial^2 P^{(1)}}{\partial \xi^2}+F_l(\tau,\xi)\,, \qquad [61]$$

where

$$B_l=-\frac{[1-u_l^{(0)}/v]}{[v^2(1-u^{(0)}/v)^2-(H-h)g]}\,.$$

From the pressure boundary condition at $y=0$, we have (for the constants of integration) the compatibility condition

$$F_l(\tau,\xi)=F_g(\tau,\xi)-(T/v^2)\frac{\partial^2 Y^{(1)}}{\partial \xi^2}\,. \qquad [62]$$

Equations [51] and [58] can be integrated and the velocity boundary conditions satisfied at $y=h$ and $-(H-h)$. The resulting expressions for the terms in the velocity boundary condition in [59] can now be substituted and a compatibility equation obtained. Using [62], the compatibility condition simplifies to

$$\hat{A}\frac{\partial^3 Y^{(1)}}{\partial \xi^3} + \hat{B} Y^{(1)}\frac{\partial Y^{(1)}}{\partial \xi} + \hat{C}\frac{\partial Y^{(1)}}{\partial \tau} = 0, \qquad [63]$$

where

$$\hat{C} = -2\left[\frac{\rho_l(v - u_l^{(0)})}{H - h} + \frac{\rho_g(v - u_g^{(0)})}{h}\right],$$

$$\hat{B} = \frac{3}{v}\left[\frac{\rho_g(v - u_g^{(0)})^2}{h^2} - \frac{\rho_l(v - u_l^{(0)})^2}{(H - h)^2}\right],$$

$$\hat{A} = \frac{1}{3v^3}[3T - \rho_g h(v - u_g^{(0)})^2 - \rho_l(H - h)(v - u_l^{(0)})^2].$$

Equation [63] is the non-linear Korteweg–deVries equation for small but finite amplitude long waves. It is immediately evident that use of [53] in [63] leads to an equation of the same form for small finite amplitude pressure waves. Note the third order dispersion term in [63], which does not arise in the averaged working equations. It contains the surface tension effect but remains even if surface tension is neglected.

The nonlinear Korteweg–deVries equation has recently been the subject of much study and many exact analytic solutions have been found. In general, transformation into a linear integral equation is possible. A consequence of the nonlinearity is the existence of waves of permanent shape (solitary waves). These are due to the nonlinearity exactly balancing the dispersion. We have developed numerical solutions to the Korteweg–deVries equation and compared these with analytical solutions where possible. The numerical solutions are now being compared with experiments in stratified flows and the results will be reported in a later paper.

*Compressible stratified flow*

For compressible flow, the conservation equations become:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i u_i) + \frac{\partial}{\partial y}(\rho_i v_i) = 0, \qquad [64]$$

$$\rho_i\frac{\partial u_i}{\partial t} + \rho_i u_i\frac{\partial u_i}{\partial x} + \rho_i v_i\frac{\partial u_i}{\partial y} = -\frac{\partial P_i}{\partial x}, \qquad [65]$$

$$\rho_i\frac{\partial v_i}{\partial t} + \rho_i u_i\frac{\partial v_i}{\partial x} + \rho_i v_i\frac{\partial v_i}{\partial y} = -\frac{\partial P_i}{\partial y} - g, \qquad [66]$$

$$\mathrm{d}P_i = C_i^2\,\mathrm{d}\rho_i. \qquad [67]$$

The last equation is the equation of state. We will also assume that $C_i$ is constant, i.e. the system is isothermal. The system has the same boundary conditions as in the incompressible case. Applying the reductive perturbation method as described previously and eliminating $\rho_i^{(o)}$ from terms not involving $\epsilon$,

$$P_i^{(0)}(y) = P_i^{(0)}(0)\exp\left(\frac{g}{C_i^2}(Y - y)\right). \qquad [68]$$

For $g/c_i^2(Y-y) \ll 1$, $P_i^{(0)}$ can be reduced to

$$P_i^{(0)}(y) = P_i^{(0)}(0) + \rho_i^0(0)g(Y-y)\,, \qquad [69]$$

which has the same form as in the incompressible case.

The process of integration proceeds in the same way as for the incompressible case, but is extremely tedious and will not be detailed. Only two interesting results are shown. First, if $V$ is the phase velocity, then the boundary conditions require

$$\gamma(V-u_g^{(0)})^2(V-u_l^{(0)})^2 + \delta(V-u_g^{(0)})^2 + \beta(V-u_l^{(0)})^2 + \epsilon = 0\,, \qquad [70]$$

where

$$\gamma = \alpha\rho_l^{(0)}C_l^2 + (1-\alpha)\rho_g^{(0)}C_g^2\,,$$

$$\delta = -(1-\alpha)[\rho_g^{(0)}C_g^2C_l^2 + \alpha C_l^2gH]\,,$$

$$\beta = -\alpha[\rho_l^{(0)}C_g^2C_l^2 - (1-\alpha)\rho_g^{(0)}C_g^2gH]\,,$$

$$\epsilon = -\alpha(1-\alpha)(\rho_l^{(0)} - \rho_g^{(0)})C_g^2C_l^2gH\,.$$

Equation [70] is identical to the characteristic polynomial obtained from the method of characteristics and the roots are given in [34] and [36]. The same stability condition also holds.

For the next order the compatibility condition simplifies to

$$\hat{A}\frac{\partial^3 Y^{(1)}}{\partial\xi^3} + \hat{B}Y^{(1)}\frac{\partial Y^{(1)}}{\partial\xi} + \hat{C}\frac{\partial Y^{(1)}}{\partial\tau} = 0\,, \qquad [71]$$

which is again the nonlinear Korteweg–deVries equation and the coefficients are:

$$\hat{A} = \frac{1}{3AV^4}(V-u_g^{(0)})(V-u_l^{(0)})[3T - \rho_g^{(0)}h(V-u_g^{(0)})^2 - \rho_l^{(0)}(H-h)(V-u_l^{(0)})^2]\,,$$

$$\hat{B} = \frac{(1-u_l^{(0)}/V)}{V(1-u_g^{(0)}/)}\Big\{(g/V + A/(\rho_g^{(0)}V))\Big[1 + \frac{\rho_g^{(0)}VC_g^2(2g/v + A/(\rho_g^{(0)}V))}{A[C_g^2 - V^2(1-u_g^{(0)}/V)^2]}$$

$$+ [AV^3(1-u_g^{(0)}/V)^4]/[\rho_g^{(0)}C_g^2[C_g^2 - V^2(1-u_g^{(0)}/V)^2]]\Big\}$$

$$-\frac{(1-u_g^{(0)}/V)}{V(1-u_l^{(0)}/v)}\Big\{(g/v + A/(\rho_l^{(0)}V))\Big[1 + \frac{\rho_l^{(0)}VC_l^2(2g/v + A/(\rho_l^{(0)}V))}{A[C_l^2 - V^2(1-u_l^{(0)}/V)^2}\Big]$$

$$+ [AV^3(1-u_l^{(0)}/V)^4]/[\rho_l^{(0)}C_l^2[C_l^2 - V^2(1-u_l^{(0)}/V)^2]]\Big\}\,,$$

$$\hat{C} = (u_g^{(0)} - u_l^{(0)})/V + \frac{(1-u_l^{(0)}/V)[1 + 2g\rho_g^{(0)}/A + V^2/C_g^2(1-u_g^{(0)}/V)^2]}{[1 - V^2/C_g^2(1-u_g^{(0)}/V)^2]}$$

$$-\frac{(1-u_g^{(0)}/V)[1 + 2g\rho_l^{(0)}/A + V^2/C_l^2(1-u_l^{(0)}/V)^2]}{[1 - V^2/C_l^2(1-u_l^{(0)}/V)^2]}\,.$$

Note that the third order dispersion term appears in [71]. These higher order dispersive effects are not seen in the averaged equations.

The governing equation for pressure waves has the same form as [71] as also obtained for incompressible flow.

The form of the equation governing finite amplitude waves in compressible flow is similar to that in incompressible flow as evident from comparing [63] and [71]. However, the coefficients are considerably more complicated. Also the phase velocity for compressible flow, $V$, has two positive values, when it is real, corresponding to right moving pressure and gravity waves. For incompressible flow the expression for phase velocity only leads to right moving gravity waves (and the associated pressure field).

An interesting result of some further algebraic manipulation ommitted here, is that the Korteweg–deVries equations for the compressible case become identical to that for the incompressible case for the values of phase velocity corresponding to gravity waves. This suggests that the gravity waves and pressure (or acoustic) waves are essentially uncoupled.

Furthermore if we neglect gravitational effects and set the initial velocities $u_l^{(0)} = u_g^{(0)} = 0$, then we obtain a form of the Korteweg–deVries equation that corresponds closely to that of Morioka & Matsui (1975). There are small differences as we have assumed a slightly different equation of state. This correspondence is to be expected since Morioka and Matsui considered pressure waves propagating into a separated gas–liquid layer at rest in a rectangular duct. They did not consider transverse body forces (like gravity) or surface tension, so their system is always unstable if there is relative mean motion between the phases.

A consequence of finite amplitude pressure waves being described by [71] is that pressure solitons may be possible. So far as we know these have not been observed experimentally in two-phase flow, but it would be interesting to look for them. There is an extensive literature on this subject as discussed in Whitham (1974), but very little has been done in two-phase flow, except for the Russian work in bubbly flow reported by Kuznetsov *et al.* (1978).

## 6. SUMMARY

Models for separated two-phase flow have been discussed with regard to averaging procedures and the treatment of pressure terms. The forms derived in this paper will be used in subsequent papers and therefore the derivation has been presented in detail. The working equations for stratified flow have been derived and analyzed to illustrate that an arbitrary interfacial configuration cannot be imposed for all flow conditions. It is shown that the stability condition for the averaged equations corresponds to that obtained by analysis of the local instantaneous formulation for long waves. A reductive perturbation method has been used for the local instantaneous formulation and will be used in subsequent papers for averaged formulations. The growth of finite amplitude waves has been shown to be described asymptotically by the Korteweg–deVries equation both for incompressible and compressible inviscid stratified flow. Thus higher order dispersive effects, that do not arise in averaged equations, at least with the assumptions used in this paper, have been identified. This leads to the possibility of waves of permanent shape arising out of the nonlinearity being balanced exactly by the dispersion. We conclude:

- In developing multiphase models it is important to recognize that the shape and motion of the interface are part of the problem being solved. An arbitrary interfacial configuration cannot be imposed without giving rise to inconsistencies within the model. For example, the flow may be stratified under the action of gravity under certain conditions, but as the gas velocity is increased, interfacial waves will develop. These waves will grow and may eventually bridge the duct giving rise to slug or plug flow. Appropriate modifications to the interfacial force term are necessary when the shape of the interface changes.
- With regard to averaging procedures time/space averaged equations are the most suitable from a practical viewpoint, but ensemble/space averaging may be needed for transients where the "signal" and "noise" are not clearly separated. Because of the expense involved in repeating sets of experiments, ensemble averaging is sometimes impractical, and more work is required for transients of this type.
- Analysis of the local instantaneous form of the conservation equations for highly

idealized interface configurations (flow regimes) may be interesting. Changes in the interface shape are taken into account naturally in such analysis and therefore show effects that do not arise in the usual forms of the averaged equations. For example, higher order dispersion terms have been identified in our analysis of stratified flow.

● In certain flow regimes, finite amplitude waves are described by non-linear equations, like the cubic Schrodinger or Korteweg–deVries equations, which have classes of exact solutions. Solutions to these equations and comparison with appropriate experiments would be of considerable interest.

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NOMENCLATURE

$A$ cross-section area
$\mathbf{a}$ arbitrary vector field
$a_i$ interfacial area per unit volume
$a_{kw}$ area of phase $k$ in contact with wall per unit volume
$B$ coefficiented of interfacial frictional term
$C_k, C'_k, C''_k$ distribution coefficients
$C_p$ specific heat
$C, c$ sound speed
$d_h$ hydraulic diameter
$E$ total energy (internal + kinetic)
$e$ specific internal energy
$F$ body force
$f$ friction factor or arbitrary function
$g$ gravitational acceleration
$H$ height of duct
$h$ distance between interface and upper boundary of duct
$h$ specific enthalpy (if subscripted with $k$)
$\bar{\bar{I}}$ identity tensor
$\mathbf{j}$ flux of conserved quantity
$\dot{m}$ mass transferred per unit volume per unit time
$\mathbf{n}$ outward drawn normal
$P$ pressure
$\mathbf{q}, q$ heat flux vector or heat transferred per unit volume per unit time
$Q$ heat generated per unit volume per unit time
$\hat{S}$ source of conserved quantity
$T$ surface tension
$t$ time
$u, U$ velocity in $z$ or $x$ direction
$v$ velocity in $y$ direction, phase velocity for incompressible flow
$\mathbf{v}, \mathbf{V}$ velocity vector
$V$ volume or phase velocity
$x, y, z$ rectangular co-ordinates
$Y$ shape of interface
$\alpha$ volume fraction
$\Gamma_m$ interfacial mass transfer
$\lambda$ characteristic direction
$\rho$ density

$\bar{\bar{\tau}}$ stress dyadic
$\tau, \xi$ stretched co-ordinates
$\tau_{wk}$ wall friction per unit volume acting on phase $k$
$\tau_{ik}$ interfacial friction per unit volume acting on phase $k$
$\psi$ conserved quantities

*Subscripts*

$1, g$ vapor phase
$2, l$ liquid phase
$i$ interface
$k$ phase index (i.e. 1 or 2)
sat saturation
$w$ wall
$z$ component in $z$ direction

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